But, since all the spheres are tangent to the given sphere, the points into which they transform are coincident, and therefore, $d'' = \delta'$. Hence

$$(k - 1/k)(r + \rho) = \left(d + \delta - \frac{4d'}{k + 1/k}\right)(k + 1/k),$$

or, $(r + \rho)/(\delta + c) = e$, where e and c are constants.

Therefore, the locus of the centers of the spheres is the locus of a point, the ratio of whose distances from the center of the given sphere to its distance from a plane parallel to and at the distance c from the common plane of similitude is constant; *i. e.*, an ellipsoid of revolution.

THE ASTROLABE.

By MARCIA LATHAM, Hunter College, New York City.

Among ancient civilized peoples for many centuries the most important instrument used by astronomers, astrologers and surveyors was the astrolabe, or, more properly, the astrolabe planisphere.

The word is derived from two Greek words meaning "to follow the stars," and is therefore applicable in general to any astronomical or astrological instrument. Indeed, the name has been applied to at least three distinct forms. The first of these, better known as the armillary sphere, or *instrumentum armillarum* (instrument of rings), consists of two, three, or more brass circles, representing the circles of the celestial sphere, hinged together in the proper relations, and bearing tubes for sighting the heavenly bodies. An engraving of this may be found on the title page of many an old book on mathematics or astronomy. It was doubtless invented by Hipparchus, and was used by Ptolemy, who described it in the *Almagest*, Book V, Proposition 1. A large wooden ring, used by mariners in the days of Columbus and Vasco da Gama to take the altitude of the sun, was also known as an astrolabe. The most important form, however, was the astrolabe planisphere, or, simply, the planisphere.

The theory of the planisphere was given in the second century of our era by Ptolemy in a treatise on the planisphere, which is not extant in the Greek but survived in the Arabic, from which it was translated by Commandinus.

The Arabs, probably applying the theory of Ptolemy, constructed exceedingly accurate astrolabes. Remains of astrolabes have been found in Assyrian excavations, and the instrument is still of practical value in India and Persia. From Arabia it passed again into Europe, and was used by astronomers and surveyors in Italy, England and other countries until the eighteenth century. Bion wrote a treatise on the astrolabe in 1702, and Boileau mentioned it in a satire written in 1693:

> "Une astrolabe en main, elle a dans sa gouttière
> À suivre Jupiter passé la nuit entière."

The simplest and most available English work on the description and use of the astrolabe is the *Tractatus de Conclusionibus Astrolabii* (*Bred and Mylke for Childeren*) written by Geoffrey Chaucer in 1391 and 1392, in the form of a letter to his "lytle sonne Lowis," then ten years old, who was already familiar with the celestial globe and had expressed a desire for information concerning the astrolabe. Chaucer's work, according to Professor Skeat, was based upon the "Compositio et Operatio Astrolabii," a Latin treatise by Messahallah, an Arabian astronomer of the ninth century.

More careful descriptions are to be found in the geometrical and astronomical treatises by Clavius (1612), Metius (1626 and 1633), Danti (1569), Bruni (1625), Reisch (1503), Gemma Frisius and Deschâles; but the standard authority is Johannes Stoefler (or Stoflerinus), whose *Elucidatio fabricæ ususque astrolabii* was published in 1510. (The copy consulted in preparing this paper was published in Oppenheim in 1524.)

The astrolabe planisphere is essentially the projection of the celestial sphere upon a plane. In most forms the projection is stereographic, that is, the point of sight is at one of the poles of the circle upon whose plane the sphere is projected. Three primitive planes have been used:

1. The plane of the equator, the point of sight being at the south pole. This form is called the "equinoctial astrolabe," and is the one suggested by Ptolemy. It is also known as the "septentrional astrolabe."
2. The plane of the equinoctial colure.
3. The plane of the solstitial colure.

The objection to the equinoctial astrolabe is that the nature of the projection varies with the latitude. To meet this difficulty Gemma Frisius used the second form, which, being independent of the location of the observer, is called the "universal astrolabe," or "Catholicon."

Johannes de Rojas, a Spaniard, projected the sphere orthographically on the plane of the solstitial colure, the result being the "Analemma," which is also a universal astrolabe. There was still the objection that in general the projections of equal arcs of a circle are not equal. To obviate this difficulty de la Hire (1640–1718) used the method of globular projection, on the plane of the meridian. In this method the point of sight is on the axis of the primitive circle, outside the sphere and at a distance from its surface equal to the product of its radius by the sine of forty-five degrees. This was further modified by Parent, but Clavius returned to the form suggested by Ptolemy, and most of his successors have followed his example.

Summarizing, then, we may say that the most important form is the equinoctial or septentrional astrolabe planisphere, suggested by Ptolemy and revived by Clavius, the principle of which is stereographic projection upon the plane of the equinoctial, the point of sight being at the south pole.

The astrolabe was usually made of brass or copper, circular in shape, and from four to seven inches in diameter (some of the Eastern instruments are much larger), and very carefully and beautifully engraved. Chardin tells us that in

Persia the astronomers themselves made them, not entrusting them to ordinary artisans.

The fundamental part is called the *mater* or mother, and is a heavy circular piece with a ring which permits it to assume a vertical position when suspended from the right thumb. One side is the back or *dorsum,* on which rotates the *rule,* a flat bar bearing sights similar to those of a gun. The other side, called the *front* or face, is hollowed out so that any one of several plates, or *tables,* may be fitted into it. Above such a plate lies the *rete* (net), and the *label,* which is simply a pointer; and the whole is fastened in place by a pin or wedge (see Fig. 1).[1]

FIG. 1.

Examining it more closely we find around the rim of the "back" several rings containing respectively the signs of the zodiac, divided into degrees; the months of the year, divided into days; and sometimes the golden letters of the church calendar, or the names of the winds. Fig. 2 is a diagram of the back. *AB* represents the meridian, and *CD* the east and west line, but it should be noted that the observer's right indicates the west and his left the east. Above the line

[1] This astrolabe is in the collection of Professor David Eugene Smith, of Columbia University.

CD are often placed the "curves of the unequal hours." These are circles, each passing through the center of the astrolabe and one of the six points of division of the quadrant, and having its center on the meridian line.

Below *CD* is the "geometric square" (*Quadratus geometricus*), or "square of the shadows," *EGHKL*. *EG, GH, HK* and *KL* are each divided into a number of equal parts. The vertical sides constitute the "*umbra versa*," which corresponds to the shadows cast on a wall by a horizontal staff fastened to the side of the wall, the altitude of the sun being not more than forty-five degrees. The horizontal part corresponds to shadows cast on the ground by a vertical staff when the sun's altitude is not less than forty-five degrees, and is called the "*umbra recta*." The geometric square is in itself a complete instrument, and was so used.

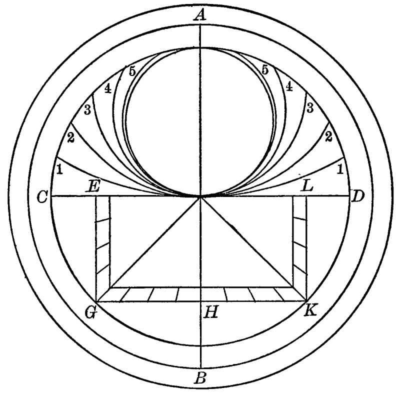

FIG. 2.

Turning the instrument over, we find that the concave part of the face bears three concentric circles, which are the projections of the equator and the tropics, the hole in the center representing the north pole.

The thin plates that fit into this depression constitute the essential part of the astrolabe, and will be described below. The rete, called by the French *l'araigne* (spider) is a plate of filigree metal, consisting of a circle representing one half of the zodiac, and many small tongues of metal, each of which serves to locate some important fixed star. Those within the zodiac have, of course, north latitude, and those without, south latitude. A point projecting from the outer rim of the rete is called the *denticle*. The wedge holding the different parts in place was often in the shape of a horse, and was named the "*equus restringens*."

With the aid of figures 3 and 4 the method of constructing the plates can be easily understood by any one familiar with the principles of stereographic projection.

In figure 3, the largest circle is the projection of the Tropic of Capricorn, *AB* that of the meridian, and *CD* of the east and west line. Arc *AH* is equal to the maximum declination of the sun. *GHM* is the projection of the equator, and *LM* that of the Tropic of Cancer. *AHMN* is therefore that of the ecliptic.

In figure 4 arc *HL* is the complement of the latitude of the place for which the plate is constructed, and *Z* is the projection of the zenith and *Y* that of the nadir. Arc *TK* is equal to arc *HL* and *M* is a point on the oblique horizon. *GMH* is the projection of the principal almucantar, or circle of altitude. Arc *LW* is equal to arc *LX* and circle *EF* is an almucantar. *GHZ* is the projection of the prime vertical. *NP* contains the centers (as *S*) of all azimuths (as *UV*), which must also pass through the zenith.

Below the horizon the plates bear certain "curves of the hours," constructed

as follows: Divide those portions of the tropics and the equator lying below the horizon each into twelve equal arcs, and pass circles through the corresponding points of division.

Lastly, since the instrument was used also for astrological purposes, the plates bear lines dividing the celestial sphere into the twelve astrological "houses." To obtain these, divide the Tropic of Capricorn into twelve equal arcs. Construct a circle through the north point of the horizon, and the points of division respectively nearest the meridian to the right above and to the left below. A similar procedure will determine all the lines separating the twelve houses.

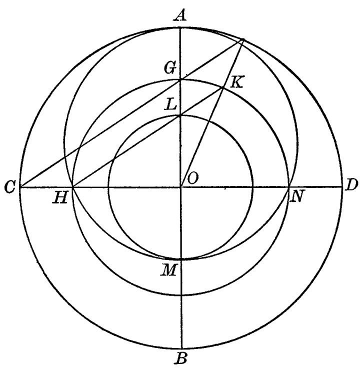

FIG. 3.

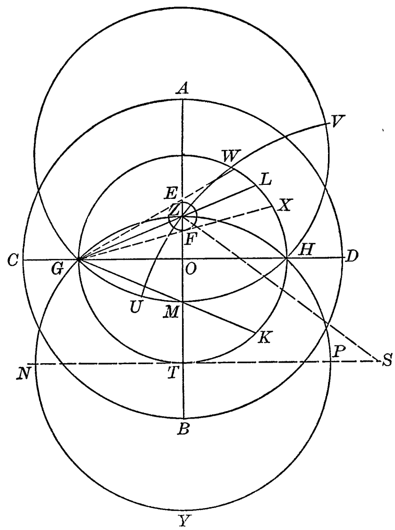

FIG. 4.

Having already explained the construction of the zodiac on the rete, it remains to determine the positions of the tongues locating the fixed stars. The position of a star can easily be fixed when its latitude and longitude are known.

In some cases, particularly among the Orientals, the instrument was constructed for only one latitude; in such instruments the sphere is projected directly upon the mother, and there are no extra plates.

A much longer article than this is to be could well be devoted to an enumeration of the uses of the astrolabe. We shall confine ourselves to mentioning a few of these.

To determine the sign and degree of the zodiac for any day of the year, use the back of the instrument, and lay the label over the day. Its extremity will indicate the corresponding degree.

The back is also used to find the altitude of the sun. Suspend the instrument on the right thumb, and turn the rule until the rays of the sun shine through both sights. The rule will then point out the number of degrees in the sun's altitude. By a similar procedure, the altitude of a star can be found.

The time of the day may be found as follows: First get the sun's altitude; turn the astrolabe over, and revolve the rete (westward or eastward according as the observation is made in the morning or afternoon) until the degree of the zodiac corresponding to that day falls upon the proper circle of altitude. Lay the label across this degree, and it will point out a number of degrees on the margin, from which the time can be calculated.

Other matters as easily determined are the time of dawn and of sunrise for a given day; the meridian altitude of the sun, given the degree of the sun; and, conversely, the degree, when the meridian altitude is known; the latitude of any place; and the cardinal points of the compass.

The curves of the unequal hours on the back may also be used to find the time of the day. Determine the meridian altitude of the sun. Place the rule on the corresponding degree and mark with ink on the rule the point where it cuts the arc of the twelfth hour. Then turn the rule until the sun shines through the sights, and the point marked on the rule will fall on or near the circle corresponding to the hour of the day.

The geometric square was invaluable to the surveyor before the invention of the telescope. The instrument known as "Jacob's Staff" was its only rival in usefulness. By means of it the surveyor could find the height of an accessible or inaccessible tower, by one or two observations; the height of a tower on an inaccessible hill; the distance between two inaccessible points; the distance between two towers not standing on the same plane; the distance of a tower whose height is known; the breadth of a stream; the depth of a pit, or of a valley with sloping walls.

Let it be required, for example, to find the height of a tower when the observer may take any position in front of it. Holding the astrolabe suspended from his right thumb, with the rule fixed in position along the diagonal of the square, the observer will walk backward or forward until he sights the top of the tower along the rule. Then the height of the tower is equal to the distance of the observer from its foot, increased by the height of the observer.

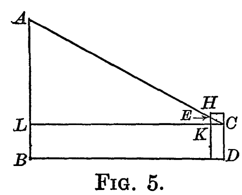

FIG. 5.

If, however, the observer must not change his position, he may move the rule around until the top of the tower is sighted, and note the point where the rule cuts the side of the square (see Fig. 5). If each side is divided into one hundred equal parts, and the rule cuts off sixty parts on the umbra versa, the height of the tower is equal to sixty hundredths of the distance of the observer from its foot (always increased by the height of the observer). If it cuts off sixty parts on the umbra recta the height is one hundred sixtieths of the distance to its foot. These statements are easily proved by means of similar triangles.

If the tower is inaccessible, two observations may be made and the ratio of the height to the distance between the two positions determined. These are some of the simplest applications, but the mathematical principles are the same throughout.

While this article does not give an exhaustive description of the construction and uses of the astrolabe, it is hoped that it may at least serve to show by what means the sciences of astronomy and surveying attained so great a degree of development before the days of Tycho Brahe and Galileo.

A USEFUL PRINCIPLE IN CURVE TRACING.

BY ARNOLD EMCH, University of Illinois.

1. The usual elementary methods of curve tracing in rectangular coördinates consist:

(1) In plotting points of the curve by assigning arbitrary values to one of the variables and finding the corresponding values of the other variable;
(2) In determining the tangents at these points by means of the derivatives;
(3) In finding the intersections of the curve with definite straight lines, or other conveniently chosen curves;
(4) In establishing possible properties of symmetry;
(5) In determining concavity and convexity, maxima and minima;
(6) In determining the asymptotes;
(7) In determining possible singular and inflexional points.

This list of tests and different steps to be taken in the investigation of a curve may, of course, be extended according to the difficulties and the nature of the problem.[1]

In ordinary curve tracing little use is made of other algebraic methods than those mentioned above. It is the purpose of this note to show the effectiveness of a certain algebra-geometric method in elementary curve tracing.

2. Restricting ourselves to algebraic curves, let such a curve be represented by the equation

$$F(x, y) = 0 \tag{1}$$

in which $F(x, y)$ is an irreducible polynomial. It is always possible to write $F(x, y) = 0$ in the identical form

$$F(x, y) \equiv \phi_1(x, y)\psi_2(x, y) - \phi_2(x, y)\psi_1(x, y) = 0, \tag{2}$$

where ϕ_1, ϕ_2, ψ_1, ψ_2 are also polynomials in x and y, of which some may reduce to constants. If it should prove to be convenient, we might use

$$F(x, y) - G(x, y) + G(x, y)$$

in place of $F(x, y)$. Thus, as $G(x, y)$ may be any polynomial, we may resolve $F(x, y)$ in an infinite number of ways into the form (2). Equation (2) is evidently the result of the elimination of λ between the simultaneous equations

[1] See PERCIVAL FROST: *An Elementary Treatise on Curve Tracing*, pp. 177–187 (1911).